4.2 Implicit Differentiation

What you will learn about . . .

- Implicitly Defined Functions
- Lenses, Tangents, and Normal Lines
- Derivatives of Higher Order
- Rational Powers of Differentiable Functions

and why . . .

Implicit differentiation allows us to find derivatives of functions that are not defined or written explicitly as a function of a single variable.

Implicitly Defined Functions

The graph of the equation $x^3 + y^3 - 9xy = 0$ (Figure 4.7) has a well-defined slope at nearly every point because it is the union of the graphs of the functions $y = f_1(x)$, $y = f_2(x)$, and $y = f_3(x)$, which are differentiable except at O and A. But how do we find the slope when we cannot conveniently solve the equation to find the functions? The answer is to treat y as a differentiable function of x and differentiate both sides of the equation with respect to x, using the differentiation rules for sums, products, and quotients, and the Chain Rule. Then solve for dy/dx in terms of x and y *together* to obtain a formula that calculates the slope at any point (x, y) on the graph from the values of x and y.

The process by which we find dy/dx is called **implicit differentiation.** The phrase derives from the fact that the equation

$$x^3 + y^3 - 9xy = 0$$

defines the functions f_1, f_2, and f_3 implicitly (i.e., hidden inside the equation), without giving us *explicit* formulas to work with.

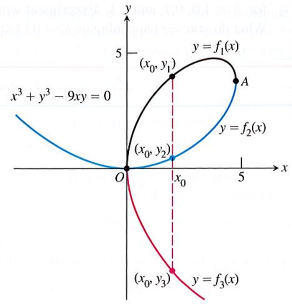


Figure 4.7 The graph of $x^3 + y^3 - 9xy = 0$ (called a *folium*). Although not the graph of a function, it is the union of the graphs of three separate functions. This particular curve dates to Descartes in 1638.

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.

SOLUTION

To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating y as a differentiable function of x and applying the Chain Rule:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1 \qquad \frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Now Try Exercise 3.

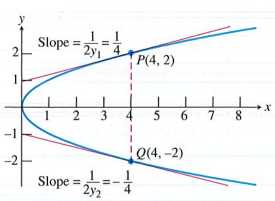


Figure 4.8 The derivative found in Example 1 gives the slope for the tangent lines at both P and Q, because it is a function of y.

In the previous example we differentiated with respect to x, and yet the derivative we obtained appeared as a function of y. Not only is this acceptable, it is actually quite useful. Figure 4.8, for example, shows that the curve has two different tangent lines when $x = 4$: one at the point $(4, 2)$ and the other at the point $(4, -2)$. Since the formula for dy/dx depends on y, our single formula gives the slope in both cases.

Implicit differentiation will frequently yield a derivative that is expressed in terms of both x and y, as in Example 2.

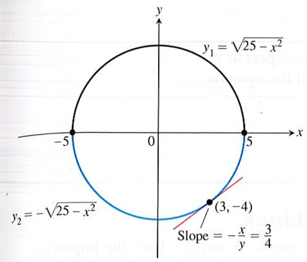


Figure 4.9 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$. (Example 2)

EXAMPLE 2 Finding Slope on a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

SOLUTION

The circle is not the graph of a single function of x, but it is the union of the graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 4.9). The point $(3, -4)$ lies on the graph of y_2, so it is possible to find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also find this slope more easily by differentiating both sides of the equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \qquad \text{Differentiate both sides with respect to } x.$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is

$$\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$$

The implicit solution, besides being computationally easier, yields a formula for dy/dx that applies at any point on the circle (except, of course, $(\pm 5, 0)$, where slope is undefined). The explicit solution derived from the formula for y_2 applies only to the lower half of the circle.

Now Try Exercise 11.

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2. We treat y as a differentiable function of x and apply the usual rules to differentiate both sides of the defining equation.

Ellen Ochoa (1958–)

After earning a doctorate degree in electrical engineering from Stanford University, Ellen Ochoa became a research engineer and, within a few years, received three patents in the field of optics. In 1990, Ochoa joined the NASA astronaut program, and, three years later, became the first Hispanic female to travel in space. Ochoa's message to young people is: "If you stay in school you have the potential to achieve what you want in the future."

EXAMPLE 3 Solving for dy/dx

Show that the slope dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

SOLUTION

First we need to know dy/dx, which we find by implicit differentiation:

$$2y = x^2 + \sin y$$

$$\frac{d}{dx}(2y) = \frac{d}{dx}(x^2 + \sin y) \qquad \text{Differentiate both sides with respect to } x \ldots$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin y)$$

$$2\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx} \qquad \ldots \text{treating } y \text{ as a function of } x \text{ and using the Chain Rule.}$$

$$2\frac{dy}{dx} - (\cos y)\frac{dy}{dx} = 2x \qquad \text{Collect terms with } dy/dx$$

$$(2 - \cos y)\frac{dy}{dx} = 2x \qquad \text{and factor out } dy/dx.$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}. \qquad \text{Solve for } dy/dx \text{ by dividing.}$$

The formula for dy/dx is defined at every point (x, y), except for those points at which $\cos y = 2$. Since $\cos y$ cannot be greater than 1, this never happens.

Now Try Exercise 13.

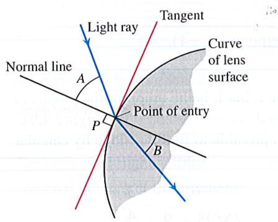


Figure 4.10 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

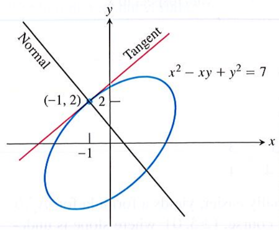


Figure 4.11 Tangent and normal lines to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (Example 4)

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Lenses, Tangents, and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 4.10). This line is called the *normal to the surface* at the point of entry. In a profile view of a lens like the one in Figure 4.10, the normal is a line perpendicular to the tangent to the profile curve at the point of entry.

Profiles of lenses are often described by quadratic curves (see Figure 4.11). When they are, we can use implicit differentiation to find the tangents and normals.

EXAMPLE 4 Tangent and normal to an ellipse

Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (See Figure 4.11.)

SOLUTION

We first use implicit differentiation to find dy/dx:

$$x^2 - xy + y^2 = 7$$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(7) \qquad \text{Differentiate both sides with respect to } x\ldots$$

$$2x - \left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) + 2y\frac{dy}{dx} = 0 \qquad \ldots\text{treating } xy \text{ as a product and } y \text{ as a function of } x.$$

$$(2y - x)\frac{dy}{dx} = y - 2x \qquad \text{Collect terms.}$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}. \qquad \text{Solve for } dy/dx.$$

We then evaluate the derivative at $x = -1,\ y = 2$ to obtain

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \left.\frac{y - 2x}{2y - x}\right|_{(-1,2)} = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{4}{5}.$$

The tangent to the curve at $(-1, 2)$ is

$$y - 2 = \frac{4}{5}(x - (-1))$$

$$y = \frac{4}{5}x + \frac{14}{5}.$$

continued

The normal to the curve at $(-1, 2)$ is

$$y - 2 = -\frac{5}{4}(x + 1)$$

$$y = -\frac{5}{4}x + \frac{3}{4}.$$

Now Try Exercise 17.

Derivatives of Higher Order

Implicit differentiation can also be used to find derivatives of higher order. Here is an example.

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

SOLUTION

To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y}, \text{ when } y \neq 0$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \text{ when } y \neq 0$$

Now Try Exercise 29.

EXPLORATION 1 An Unexpected Derivative

Consider the set of all points (x, y) satisfying the equation $x^2 - 2xy + y^2 = 4$. What does the graph of the equation look like? You can find out in two ways in this Exploration.

1. Use implicit differentiation to find dy/dx. Are you surprised by this derivative?
2. Knowing the derivative, what do you conjecture about the graph?
3. What are the possible values of y when $x = 0$? Does this information enable you to refine your conjecture about the graph?
4. The original equation can be written as $(x - y)^2 - 4 = 0$. By factoring the expression on the left, write two equations whose graphs combine to give the graph of the original equation. Then sketch the graph.
5. Explain why your graph is consistent with the derivative found in part 1.

Rational Powers of Differentiable Functions

We know that the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

holds for any integer n (Rules 2 and 7). We can now prove that it holds when n is any rational number.

RULE 9 Power Rule for Rational Powers of x

If n is any rational number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $n < 1$, then the derivative does not exist at $x = 0$.

Proof Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then

$$y^q = x^p.$$

Since p and q are integers (for which we already have the Power Rule), we can differentiate both sides of the equation with respect to x and obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

If $y \neq 0$, we can divide both sides of the equation by qy^{q-1} to solve for dy/dx, obtaining

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$$

$$= \frac{p}{q}\cdot\frac{x^{p-1}}{(x^{p/q})^{q-1}} \qquad y = x^{p/q}$$

$$= \frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-p/q}} \qquad \frac{p}{q}(q-1) = p - \frac{p}{q}$$

$$= \frac{p}{q}\cdot x^{(p-1)-(p-p/q)} \qquad \text{A law of exponents}$$

$$= \frac{p}{q}\cdot x^{(p/q)-1}.$$

This proves the rule. ■

By combining this result with the Chain Rule, we get an extension of the Power Chain Rule to rational powers of u:

If n is a rational number and u is a differentiable function of x, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx},$$

provided that $u \neq 0$ if $n < 1$.

The restriction that $u \neq 0$ when $n < 1$ is necessary because 0 might be in the domain of u^n but not in the domain of u^{n-1}, as we see in the first two parts of Example 6.

EXAMPLE 6 Using the Rational Power Rule

(a) $\dfrac{d}{dx}(\sqrt{x}) = \dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

Notice that $\sqrt{x}$ is defined at $x = 0$, but $1/(2\sqrt{x})$ is not.

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}(x^{-1/3}) = \dfrac{2}{3x^{1/3}}$

The original function is defined for all real numbers, but the derivative is undefined at $x = 0$. Recall Figure 3.12, which showed that this function's graph has a *cusp* at $x = 0$.

(c) $\dfrac{d}{dx}(\cos x)^{-1/5} = -\dfrac{1}{5}(\cos x)^{-6/5}\cdot\dfrac{d}{dx}(\cos x)$

$= -\dfrac{1}{5}(\cos x)^{-6/5}(-\sin x)$

$= \dfrac{1}{5}\sin x(\cos x)^{-6/5}$

Now Try Exercise 33.

Quick Review 4.2 *(For help, go to Section 1.2 and Appendix A.5.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$
2. $4x^2 + 9y^2 = 36$
3. $x^2 - 4y^2 = 0$
4. $x^2 + y^2 = 9$
5. $x^2 + y^2 = 2x + 3$

In Exercises 6–8, solve for y' in terms of y and x.

6. $x^2y' - 2xy = 4x - y$
7. $y'\sin x - x\cos x = xy' + y$
8. $x(y^2 - y') = y'(x^2 - y)$

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

9. $\sqrt{x}(x - \sqrt[3]{x})$
10. $\dfrac{x + \sqrt[3]{x^2}}{\sqrt{x^3}}$

Section 4.2 Exercises

In Exercises 1–8, find dy/dx.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $y^2 = \dfrac{x-1}{x+1}$
4. $x^2 = \dfrac{x-y}{x+y}$
5. $x = \tan y$
6. $x = \sin y$
7. $x + \tan(xy) = 0$
8. $x + \sin y = xy$

In Exercises 9–12, find dy/dx and find the slope of the curve at the indicated point.

9. $x^2 + y^2 = 13, \quad (-2, 3)$
10. $x^2 + y^2 = 9, \quad (0, 3)$
11. $(x-1)^2 + (y-1)^2 = 13, \quad (3, 4)$
12. $(x+2)^2 + (y+3)^2 = 25, \ (1, -7)$

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$
14. $x = \cos y$
15. $x^3 + y^3 = xy$
16. $x^2 + 4xy + 4y^2 - 3x = 6$

In Exercises 17–26, find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

17. $x^2 + xy - y^2 = 1, \quad (2, 3)$
18. $x^2 + y^2 = 25, \quad (3, -4)$
19. $x^2y^2 = 9, \ (-1, 3)$
20. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$
21. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$
22. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$
23. $2xy + \pi\sin y = 2\pi, \quad (1, \pi/2)$
24. $x\sin 2y = y\cos 2x, \quad (\pi/4, \pi/2)$
25. $y = 2\sin(\pi x - y), \quad (1, 0)$
26. $x^2\cos^2 y - \sin y = 0, \quad (0, \pi)$

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

27. $x^2 + y^2 = 1$
28. $x^{2/3} + y^{2/3} = 1$
29. $y^2 = x^2 + 2x$
30. $y^2 + 2y = 2x + 1$

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$
32. $y = x^{-3/5}$
33. $y = \sqrt[3]{x}$
34. $y = \sqrt[4]{x}$
35. $y = (2x + 5)^{-1/2}$
36. $y = (1 - 6x)^{2/3}$
37. $y = x\sqrt{x^2 + 1}$
38. $y = \dfrac{x}{\sqrt{x^2 + 1}}$
39. $y = \sqrt{1 - \sqrt{x}}$
40. $y = 3(2x^{-1/2} + 1)^{-1/3}$
41. $y = 3(\csc x)^{3/2}$
42. $y = [\sin(x + 5)]^{5/4}$

43. Which of the following could be true if $f''(x) = x^{-1/3}$?

(a) $f(x) = \frac{3}{2}x^{2/3} - 3$ (b) $f(x) = \frac{9}{10}x^{5/3} - 7$

(c) $f'''(x) = -\frac{1}{3}x^{-4/3}$ (d) $f'(x) = \frac{3}{2}x^{2/3} + 6$

44. Which of the following could be true if $g''(t) = 1/t^{3/4}$?

(a) $g'(t) = 4\sqrt[4]{t} - 4$ (b) $g'''(t) = -4/\sqrt[4]{t}$

(c) $g(t) = t - 7 + (16/5)t^{5/4}$ (d) $g'(t) = (1/4)t^{1/4}$

45. *The Eight Curve* (a) Find the slopes of the figure-eight-shaped curve

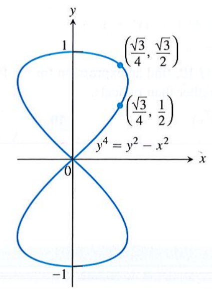
$$y^4 = y^2 - x^2$$

at the two points shown on the graph that follows.

(b) Use parametric mode and the two pairs of parametric equations

$$x_1(t) = \sqrt{t^2 - t^4}, \quad y_1(t) = t,$$
$$x_2(t) = -\sqrt{t^2 - t^4}, \quad y_2(t) = t,$$

to graph the curve. Specify a window and a parameter interval.



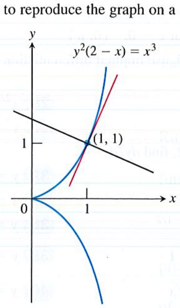
46. *The Cissoid of Diocles (dates from about 200 B.C.E.)*

(a) Find equations for the tangent and normal to the cissoid of Diocles,

$$y^2(2 - x) = x^3,$$

at the point (1, 1) as pictured below.

(b) Explain how to reproduce the graph on a grapher.



47. (a) Confirm that $(-1, 1)$ is on the curve defined by $x^3y^2 = \cos(\pi y)$.

(b) Use part (a) to find the slope of the line tangent to the curve at $(-1, 1)$.

48. Group Activity

(a) Show that the relation

$$y^3 - xy = -1$$

cannot be a function of x by showing that there is more than one possible y-value when $x = 2$.

(b) On a small enough square with center (2, 1), the part of the graph of the relation within the square will define a function $y = f(x)$. For this function, find $f'(2)$ and $f''(2)$.

49. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

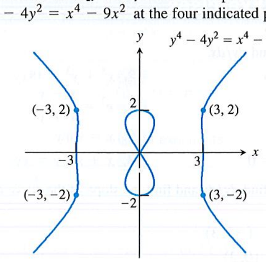
50. Find points on the curve $x^2 + xy + y^2 = 7$ **(a)** where the tangent is parallel to the x-axis and **(b)** where the tangent is parallel to the y-axis. (In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?)

51. *Orthogonal Curves* Two curves are *orthogonal* at a point of intersection if their tangents at that point cross at right angles. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal at (1, 1) and (1, −1). Use parametric mode to draw the curves and to show the tangent lines.

52. The position of a body moving along a coordinate line at time t is $s = (4 + 6t)^{3/2}$, with s in meters and t in seconds. Find the body's velocity and acceleration when $t = 2$ sec.

53. The velocity of a falling body is $v = 8\sqrt{s - t} + 1$ feet per second at the instant t (sec) the body has fallen s feet from its starting point. Show that the body's acceleration is 32 ft/sec².

54. *The Devil's Curve (Gabriel Cramer [the Cramer of Cramer's Rule], 1750)* Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



55. *The Folium of Descartes* (See Figure 4.7 on page 162)

(a) Find the slope of the folium of Descartes, $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

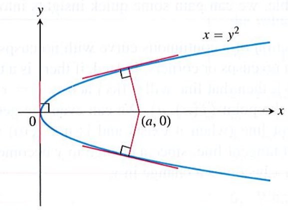
(b) At what point other than the origin does the folium have a horizontal tangent?

(c) Find the coordinates of point A in Figure 4.7, where the folium has a vertical tangent.

56. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1) intersects the curve at what other point?

57. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

58. Show that if it is possible to draw these three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown here, then a must be greater than 1/2. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



Standardized Test Questions

59. True or False The slope of $xy^2 + x = 1$ at (1/2, 1) is 2. Justify your answer.

60. True or False The derivative of $y = \sqrt[3]{x}$ is $\frac{1}{3x^{2/3}}$. Justify your answer.

In Exercises 61 and 62, use the curve $x^2 - xy + y^2 = 1$.

61. Multiple Choice Which of the following is equal to dy/dx?

(A) $\frac{y - 2x}{2y - x}$ (B) $\frac{y + 2x}{2y - x}$ (C) $\frac{2x}{x - 2y}$

(D) $\frac{2x + y}{x - 2y}$ (E) $\frac{y + 2x}{x}$

62. Multiple Choice Which of the following is equal to $\frac{d^2y}{dx^2}$?

(A) $-\frac{6}{(2y - x)^3}$ (B) $\frac{10y^2 - 10x^2 - 10xy}{(2y - x)^3}$

(C) $\frac{8x^2 - 4xy + 8y^2}{(x - 2y)^3}$ (D) $\frac{10x^2 + 10y^2}{(x - 2y)^3}$ (E) $\frac{2}{x}$

63. Multiple Choice Which of the following is equal to dy/dx if $y = x^{3/4}$?

(A) $\frac{3x^{1/3}}{4}$ (B) $\frac{4x^{1/4}}{3}$ (C) $\frac{3x^{1/4}}{4}$ (D) $\frac{4}{3x^{1/4}}$ (E) $\frac{3}{4x^{1/4}}$

64. Multiple Choice Which of the following is equal to the slope of the tangent to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$?

(A) $-\frac{1}{\sqrt{2}}$ (B) $-\sqrt{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$ (E) 0

Extending the Ideas

65. *Finding Tangents*

(a) Show that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1) has equation

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.$$

(b) Find an equation for the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_1, y_1).

66. *End Behavior Model* Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Show that

(a) $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$.

(b) $g(x) = (b/a)|x|$ is an end behavior model for $f(x) = (b/a)\sqrt{x^2 - a^2}$.

(c) $g(x) = -(b/a)|x|$ is an end behavior model for $f(x) = -(b/a)\sqrt{x^2 - a^2}$.

Quick Quiz for AP* Preparation: Sections 4.1–4.2

1. Multiple Choice Which of the following gives $\frac{dy}{dx}$ for $y = \sin^4(3x)$?

(A) $4\sin^3(3x)\cos(3x)$ (B) $12\sin^3(3x)\cos(3x)$

(C) $12\sin(3x)\cos(3x)$ (D) $12\sin^3(3x)$

(E) $-12\sin^3(3x)\cos(3x)$

2. Multiple Choice What is the slope of the line tangent to the curve $2x^2 - 3y^2 = 2xy - 6$ at the point (3, 2)?

(A) 0 (B) $\frac{4}{9}$ (C) $\frac{7}{9}$ (D) $\frac{6}{7}$ (E) $\frac{5}{3}$

3. Multiple Choice Which of the following gives $\frac{dy}{dx}$ for the parametric curve $x = 3\sin t$, $y = 2\cos t$?

(A) $-\frac{3}{2}\cos t$ (B) $\frac{3}{2}\cos t$ (C) $-\frac{2}{3}\tan t$

(D) $\frac{2}{3}\tan t$ (E) $\tan t$

4. Free Response A curve in the xy-plane is defined by $xy^2 - x^3y = 6$.

(a) Find $\frac{dy}{dx}$.

(b) Find an equation for the tangent line at each point on the curve with x-coordinate 1.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.